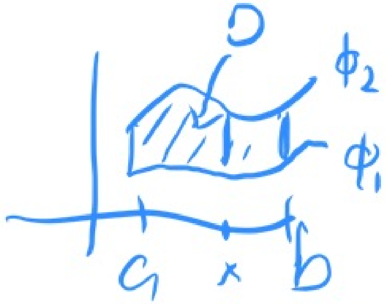
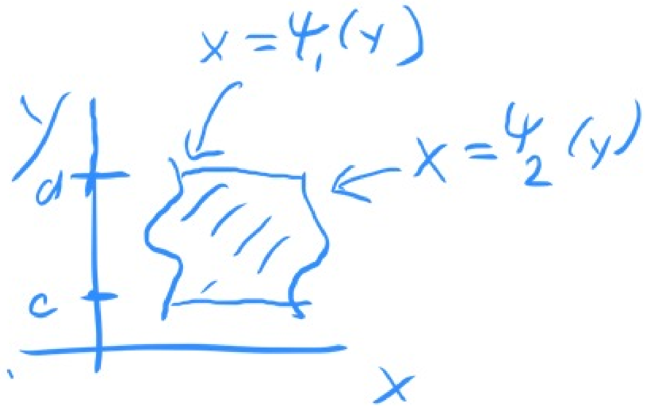
Notion of x-simple / y-simple region

Def. region $D \subset \mathbb{R}^2$ called <u>y-simple</u>

"if region is between graphs of functions $\phi_1(x)$ and $\phi_2(x)$"

precise: If $\exists\, a < b$ numbers and functions $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ such that $D = \{(x,y),\ a \leq x \leq b,\ \phi_1(x) \leq y \leq \phi_2(x)\}$



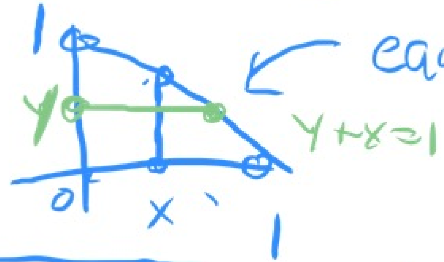


x-simple:

Many regions are both x-simple and y-simple

Examples

ⓐ D = triangle with corners $(0,0)$, $(0,1)$, $(1,0)$



equation of line: $x+y=1 \Rightarrow \boxed{y=1-x}$

Equ. of line.

$\boxed{\text{y-simple:}}$ all possible values for x:

$$0 \leq x \leq 1$$

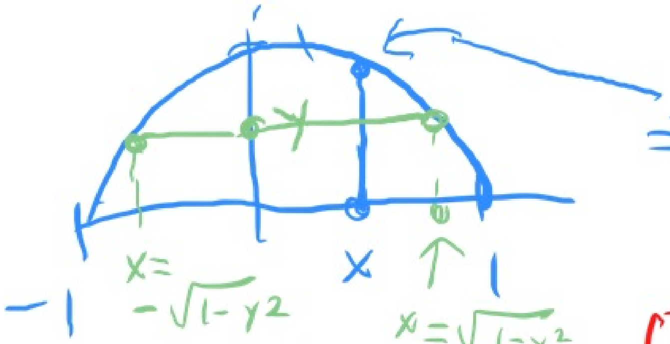
for given x: $0 \leq y \leq 1-x$

$\boxed{\text{x-simple}}$ all possible values for y:

$$0 \leq y \leq 1$$

for given y: $0 \leq x \leq 1-y$

(b) D upper semidish of radius 1



$x^2 + y^2 = 1$

$\Rightarrow y = \sqrt{1-x^2}$

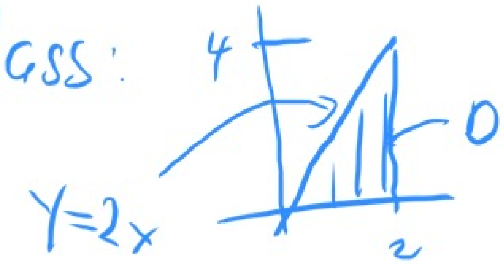
y-simple: possible x-values: $\boxed{-1 \leq x \leq 1}$

for given x " y-values: $\boxed{0 \leq y \leq \sqrt{1-x^2}}$

x-simple: possible y-values $\boxed{0 \leq y \leq 1}$

for given y: " x-values $\boxed{-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}}$

(c) last class:



$\iff x = y/2$

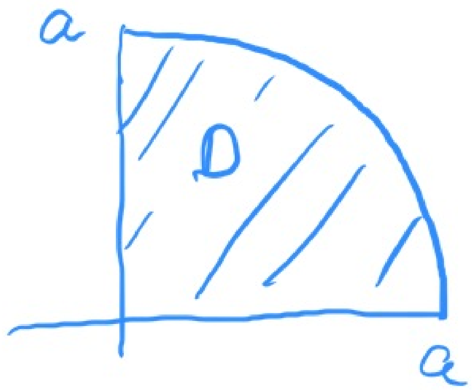
$$\boxed{\begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 2x \end{array}} \quad \text{y-simple}$$

$$\boxed{\begin{array}{l} 0 \le y \le 4 \\ y/2 \le x \le 2 \end{array}} \quad \text{x-simple}$$

Example: Calculate the integral

$$\underbrace{\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-y^2}\, dy}\, dx$$

$$\int \sqrt{a^2-y^2}\, dy = ?$$

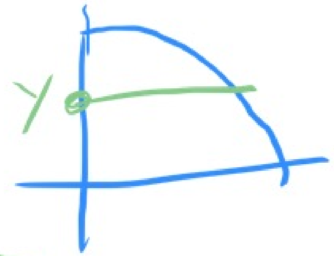


Key observation: region is both x-simple and y-simple.

Can also express D as

$$0 \le y \le a$$

$$0 \le x \le \sqrt{a^2-y^2}$$



← limit different

$$\Rightarrow \text{integral} = \boxed{\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2}\, dx\, dy}$$

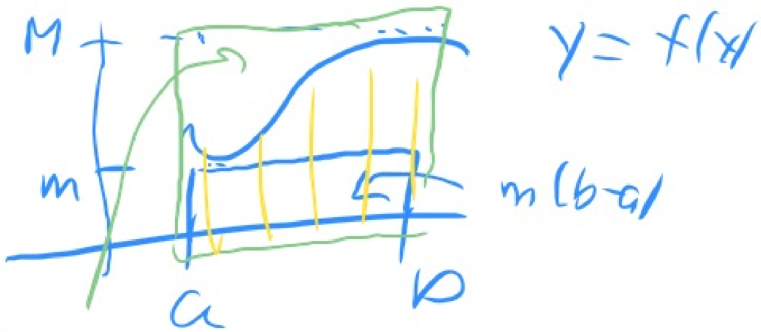
dx · dy in different order as in previous page

$$= \int_0^a \sqrt{a^2-y^2}\, x \Big|_{x=0}^{x=\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \sqrt{a^2-y^2} \cdot \sqrt{a^2-y^2}\, dy = \int_0^a a^2-y^2\, dy$$

Mean Value Inequality:

1-dim case: assume we have numbers m, M such that $m \le f(x) \le M$ for $a \le x \le b$



$$\Rightarrow \boxed{m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)}$$

yellow area

green area

area = M(b-a)

2-dim case

$f: D \subset \mathbb{R}^2 \to \mathbb{R}$, assume $m \le f(x,y) \le M$ for all (x,y) in D

$$\Rightarrow \boxed{m\ \text{area}(D) \le \iint_D f(x,y)\,dA \le M\ \text{area}(D)}$$

<u>Example:</u> Let $f(x,y) = \dfrac{1}{\sqrt{1+2x^8+y^{12}}}$

Let $D = [0,1] \times [0,1]$

use mean value inequality to estimate $\iint_D f(x,y)\,dA$

<u>Sol:</u> smallest possible value of f in D

for largest possible value in denominator

$$1+2\underbrace{x^8}_{|x|\le 1}+\underbrace{y^{12}}_{|y|\le 1} \le 1+2\cdot 1^8+1^{12} = 4$$

$\Rightarrow$ $m = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$ smallest value

largest value $\Leftrightarrow$ smallest value of denom. at $\begin{matrix} x=0 \\ y=0 \end{matrix}$

$M = \dfrac{1}{\sqrt{1}} = 1$